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THE STABLE GEOMETRIC DIMENSION OF VECTOR BUNDLES OVER REAL PROJECTIVE SPACES

BY

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ABSTRACT. An elementary argument shows that the geometric dimension of any vector bundle of order 2^e over RP^n depends only on e and the residue of n mod 8 for n sufficiently large. In this paper we calculate this geometric dimension, which is approximately $2e$. The nonlifting results are easily obtained using the spectrum bJ. The lifting results require bo-resolutions. Half of the paper is devoted to proving Mahowald's theorem that beginning with the second stage bo-resolutions act almost like $K(Z_2)$-resolutions.

1. Introduction. Let P_k^l denote the stunted real projective RP^l/RP^{k-1}. In the stable range $(l + 8 < 2k - 1)$ there is a map $P_{k+8}^{l+8} \to^{\varphi} P_k^l$ (when k is odd and l even) which induces an isomorphism in KO-theory. Thus if f: $P_k^l \to BO$ classifies a (stable) vector bundle of order 2^e in $(KO)\tilde{}(P^l)$, then $f\varphi$ classifies a bundle of order 2^e in $(KO)\tilde{}(P^{l+8})$. Since the geometric dimension $\text{gd}(\theta)$ of a stable vector bundle θ is the smallest r such that its classifying map lifts to BO_r, this implies that for fixed e the gd of a bundle of order 2^e over P^n is a nonincreasing function of n for n in a fixed mod 8 congruence class, and hence must achieve a stable value. In this paper we compute this stable value for almost every e.

THEOREM 1.1. (i) *If $e \geq 20$ and n is sufficiently large, the geometric dimension of any vector bundle of order 2^e over P^n is $2e + \delta(n, e)$, where δ is given by*

		n mod 8: 0	1	2	3	4	5	6	7
e mod 4	0	0	0	0	0	0	0	0	0
	1	2	0	0	0	0	0	2	2
	2	2	0	−1	−1	−1	0	2	2
	3	1	0	−2	−2	−2	0	1	1

(ii) *For any e and n the geometric dimension of any vector bundle of order 2^e over P^n is $\geq 2e + \delta(n, e)$.*

1.1(i) can be proved by these methods for some smaller values of e (see Remark 4.3), but for some very small values of e, the stable geometric dimension is greater

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than $2e + \delta(n, e)$. The condition "n sufficiently large" can be made precise as $n \geqslant 4e + 30 + 4[\log_2(e + 4)]$.

The nonlifting results ($\geqslant$ part) are virtually identical to those of every other approach to the geometric dimension question which utilizes, however indirectly, the Adams operation $\psi^3 - 1$ ([4], [9], [6], [7]). In certain congruences our results are slightly stronger because the results were overlooked in [4], and [9] did not consider n odd.

The significance of our work is that these bJ-primary obstructions are a total obstruction for the lifting questions dealt with in 1.1(i). It shows that our control over homotopy theory as applied to this classical geometrical question is rather complete.

REMARK 1.2. Another interpretation of 1.1(i) is that for these bundles the geometric dimension equals the fibre-homotopy geometric dimension (see Remark 2.6).

The most significant and most novel part of the theorem is the lifting results. This represents the first successful attempt to use obstruction theory with arbitrarily many nontrivial obstruction groups. Since the obstruction groups, i.e. homotopy groups of Stiefel manifolds, are known completely only through at most 29 nontrivial dimensions [13], a lifting procedure which works for purely dimensional reasons is clearly required here. This is provided by Mahowald's theory of bo-resolutions [14]. In §3 we expand considerably upon the proof given in [14], adapting the result to spaces other than S°.

Another novel aspect of our lifting results is that we are usually lifting far beyond the metastable range. For example, the case $e = 20$, $n = 8l + 4$, says $\operatorname{gd}(a \cdot 2^{4l-17}\xi_{8l+4}) = 40$, when a is odd and $l \geqslant 16$. Thus a theory of unstable resolutions is required; this is provided by the λ-algebera methods of Curtis et al.

2. Nonlifting results.

Let $\nu(m)$ denote the exponent of the largest 2-power dividing m. In this section we shall prove

THEOREM 2.1. *If* $n \leqslant 4\nu(m) - 3$, *then* $\operatorname{gd}(m\xi_n) \geqslant n - 2\nu(m) + \varepsilon(n, \nu(m))$ *where* ε *is given by the table*

		n mod 4: 0	1	2	3
$\nu(m)$ mod 4	0	0	1	1	0
	1	1	1	2	1
	2	2	1	2	1
	3	2	1	0	−1

Since the order of $m\xi_n$ in $(KO)\tilde{}(P^n)$ is $2^{\varphi(n)-\nu(m)}$, where $\varphi(n)$ is the number of positive integers $\leqslant n$ which are $\equiv 0, 1, 2,$ or 4 (8), Theorem 2.1 readily implies that the numbers in 1.1 are lower bounds.

Let $V_r = SO/SO_r$ and $P_r = RP^\infty/RP^{r-1}$. Let $QX = \Omega^\infty\Sigma^\infty X$. The functor Q may be considered as the composite of the functor Σ^∞: Spaces $\to$ Spectra which forms the suspension spectrum with the functor Ω^∞: Spectra $\to$ Spaces which takes the 0th space in the associated Ω-spectrum. The inclusion $P_r \to V_r$ admits a stable retraction $V_r \to QP_r$. This follows from early work of James. See, for example, [**19**, ¶4.2].

Let bo denote the spectrum for connective KO-theory localized at 2, $\Sigma^4 bsp$ the spectrum obtained from bo by killing π_i for $i < 4$, and bJ the fibre of the Adams operation $\theta = \psi^3 - 1$: $bo \to \Sigma^4 bsp$ [**16**], [**5**]. The map $\Sigma^\infty P_r \to P_r \wedge bJ$ induces a map $QP_r \to \Omega^\infty(P_r \wedge bJ)$.

If $n \leqslant 4\nu(m) - 3$ (the hypothesis of 2.1), $m\xi_n$ is trivial on a skeleton of dimension $b - 1 > n/2$, so that its classifying map f_m factors through P_b^n, which has a desuspension $\Sigma^{-1}P_b^n$. Adjoint to f is a map $\Sigma^{-1}P_b^n \to^{m\xi} SO$. 2.1 will be proved by showing that if $r = n - 2\nu(m) + \varepsilon - 1$, the composite

$$\Sigma^{-1}P_b^n \xrightarrow{m\xi} SO \to SO/SO_r = V_r \to QP_r \to \Omega^\infty(P_r \wedge bJ) \tag{2.2}$$

is essential.

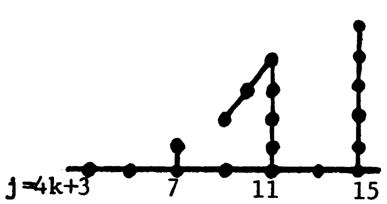
If X is a space and b is a spectrum, $[X, b]$ will denote $[\Sigma^\infty X, b] \approx [X, \Omega^\infty b]$. In the exact sequence below, the isomorphism $[\Sigma^{-1}X, \Sigma^{-1}b] \approx [X, b]$ has been used in the first two groups.

$$\begin{aligned}&\left[P_b^n, P_r \wedge bo\right] \xrightarrow{\theta_*} \left[P_b^n, P_r \wedge \Sigma^4 bsp\right]\\ &\qquad \to \left[\Sigma^{-1}P_b^n, P_r \wedge bJ\right] \to \left[\Sigma^{-1}P_b^n, P_r \wedge bo\right].\end{aligned} \tag{2.3}$$

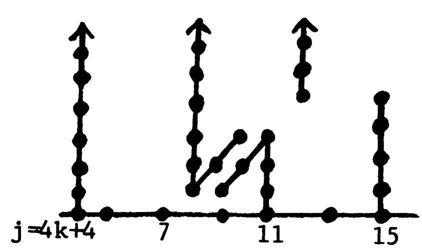
The groups $\pi_*(P_r \wedge \Sigma^4 bsp)$ have been calculated, e.g. in [**6**, 3.4], to be given by charts which begin as below and continue similarly. In these charts dots indicate nonzero elements, vertical lines multiplication by 2, diagonal lines multiplication by $\eta \in \pi_1(S^0)$, and an arrow an infinite tower. Thus for example

$$\pi_{4k+11}(P_{4k-1} \wedge \Sigma^4 bsp) \approx \mathbf{Z}_{32} \quad \text{and} \quad \pi_{4k+4}(P_{4k} \wedge \Sigma^4 bsp) \approx \mathbf{Z}_{(2)},$$

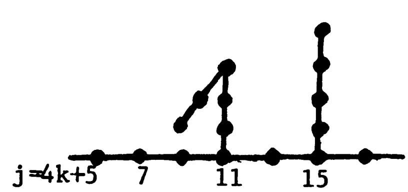
the integers localized at 2.



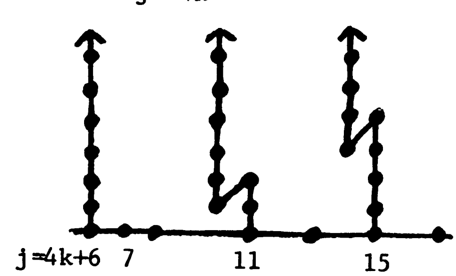
$\pi_j(P_{4k-1} \wedge \Sigma^4 bsp)$



$\pi_j(P_{4k} \wedge \Sigma^4 bsp)$



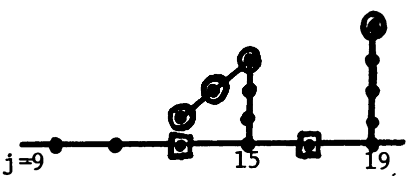
$\pi_j(P_{4k+1} \wedge \Sigma^4 bsp)$



$\pi_j(P_{4k+2} \wedge \Sigma^4 bsp)$

We describe $[P_b^n, P_r \wedge \Sigma^4 bsp]$ below in terms of the above charts, in the case b odd, which can always be arranged. It is a sum of $\mathbf{Z}_2$'s of filtration 0 plus the cyclic groups described below. If $n \equiv 3$ (4), $[P_b^n, P_r \wedge \Sigma^4 bsp] \approx [P_b^{n-1}, P_r \wedge \Sigma^4 bsp] + \pi_n(P_r \wedge \Sigma^4 bsp)$, so for the rest of the paragraph we shall assume $n \not\equiv 3$ (4). If r is odd there is one cyclic summand, with a class for each element of maximal filtration in $\pi_j(P_r \wedge \Sigma^4 bsp)$ with $j \equiv 1, 2, 3, 7$ (8) and $b < j \leqslant n$. Note that this puts one element in each filtration through a range of filtrations. If $r \equiv 2$ (4), there is one cyclic summand, with a class for an element of maximal filtration in $\pi_j(P_r \wedge \Sigma^4 bsp)$ for each $j \equiv 3$ (4) with $b \leqslant j < n$ and a class for an element of minimal filtration in $\pi_j(P_r \wedge \Sigma^4 bsp)$ with $j \equiv 2$ (4) and $b < j \leqslant n$. If $r \equiv O(4)$, $P_r \wedge \Sigma^4 bsp \simeq S^r \wedge \Sigma^4 bsp \vee P_{r+1} \wedge \Sigma^4 bsp$. Each part of this splitting contributes a cyclic summand, the latter of which was described above. There is a summand with a class for an element of minimal filtration in each $\pi_j(S^r \wedge \Sigma^4 bsp)$ with $j \equiv 0, 1, 2, 4$ (8) and $b \leqslant j \leqslant n$.

For example, $[P_{13}^{20}, P_5 \wedge \Sigma^4 bsp] \approx \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_{16}$ with the filtration zero $\mathbf{Z}_2$'s corresponding to the elements in boxes in the chart below, and the $\mathbf{Z}_{16}$, generated by a filtration 1 class, corresponding to the classes circled in the chart below.



$\pi_j(P_5 \wedge \Sigma^4 bsp)$

By similar methods one shows that $[\Sigma^{-1} P_b^n, P_r \wedge bo]$ is a sum of filtration zero $\mathbf{Z}_2$'s. If $n \not\equiv 3$ (4) and $r \not\equiv 0$ (4), the homomorphism θ_* in (2.3) is zero by [**6**, p. 152], and hence above filtration zero $[P_b^n, P_r \wedge \Sigma^4 bsp] \to [\Sigma^{-1} P_b^n, P_r \wedge bJ]$ is an isomorphism which increases filtration by 1. The element of order 2 in the additional summand of $[P_b^n, P_r \wedge \Sigma^4 bsp]$ when $n \equiv 3$ (4) or $r \equiv 0$ (4) is hit by θ_* unless $\nu(n+1) \geqslant \varphi(n-r)$ or $\nu(r) \geqslant \varphi(n-r)$.

PROPOSITION 2.4. *The composite* (2.2) *is essential if and only if there is a nonzero class of filtration* $\nu(m) - 3$ *in*

$$\begin{cases} [P_b^{n-1}, P_r \wedge \Sigma^4 bsp] & \text{if } n \equiv 3\ (4) \text{ and } \nu(n+1) < \varphi(n-r), \\ [P_b^n, P_{r+1} \wedge \Sigma^4 bsp] & \text{if } r \equiv 0\ (4) \text{ and } \nu(r) < \varphi(n-r), \\ [P_b^n, P_r \wedge \Sigma^4 bsp] & \text{otherwise.} \end{cases}$$

PROOF.

$$\begin{aligned} (KO)^{\sim}(P_b^n) &\approx [\Sigma^{-1} P_b^n, SO] \to [\Sigma^{-1} P_b^n, Q(SO)] \\ &\to [\Sigma^{-1} P_b^n, Q(P_1)] \to [\Sigma^{-1} P_b^n, P_1 \wedge bJ] \\ &\approx [P_b^n, P_1 \wedge \Sigma^4 bsp] \to [P_b^n, P_r \wedge \Sigma^4 bsp] \end{aligned}$$

are all homomorphisms, since the homotopy and Whitney sums must agree on $[\ ,Q(SO)]$. Thus divisibility by 2 in $(KO)\tilde{\ }(P^n)$ corresponds to filtration in $[P^n, P_r \wedge \Sigma^4 bsp]$.

If a is any odd integer, $16a\xi_8$ is trivial and $16a\xi_9$ is nontrivial but not detected by Stieffel-Whitney classes. Thus the element of $[P^n, P_1 \wedge \Sigma^4 bsp]$ corresponding to $16a\xi$ is trivial on P^8 and detected on the 9-cell by the filtration 1 class in $\pi_9(P_1 \wedge \Sigma^4 bsp)$. Thus by additivity $2^e a\xi$ corresponds to a class of filtration $e - 3$.

When $n \equiv 3$ (4), the class of $2^e a\xi$ in $[P^n, P_r \wedge \Sigma^4 bsp]$ is the sum of the filtration $e - 3$ classes in both summands. This follows since the map extends over P^{n+1}, and $[P^{n+1}, P_r \wedge \Sigma^4 bsp] \to [P^n, P_r \wedge \Sigma^4 bsp]$ maps to the sums. This is important because the $\pi_n(P_r \wedge \Sigma^4 bsp)$-summand extends one filtration higher than $[P^{n-1}, P_r \wedge \Sigma^4 bsp]$, giving stronger results if the top class is not in the image of θ_*.

Similarly if $r \equiv 0$ (4), $[P^n, P_r \wedge \Sigma^4 bsp]$ has a summand due to the r-cell which extends one filtration higher than its $[P^n, P_{r+1} \wedge \Sigma^4 bsp]$-summand. This is utilized effectively if it is not in the image of θ_* because $[P^n, P_1 \wedge \Sigma^4 bsp] \to [P^n, P_r \wedge \Sigma^4 bsp]$ maps onto the sum of the two summands when $r \equiv 0$ (4). □

From the charts of $\pi_*(P_r \wedge \Sigma^4 bsp)$ and the discussion of how to determine $[P^n, P_r \wedge \Sigma^4 bsp]$ from them, we tabulate

PROPOSITION 2.5. *If $n \not\equiv 3$ (4), the largest $r \not\equiv 0$ (4) such that $[P^n, P_r \wedge \Sigma^4 bsp]$ contains a nonzero class in filtration $e - 3$ is $n - 2e + \varepsilon(n, e) - 1$, where ε is as in* 2.1.

2.1 follows immediately from 2.4 and 2.5.

REMARK 2.6. Let SG_r denote the space of degree 1 maps $S^{r-1} \to S^{r-1}$ and $SG = \bigcup_r SG_r$. BSG_r is the classifying space for fiber homotopy equivalence classes of S^{r-1}-fibrations. There is a map of fibrations

$$\begin{array}{ccccccccc} SO_r & \to & SO & \to & V_r & \to & BSO_r & \to & BSO \\ \downarrow & & \downarrow & & \downarrow & & \downarrow j_r & & \downarrow j \\ SG_r & \to & SG & \to & SG/SG_r & \to & BSG_r & \to & BSG \end{array}$$

in which j corresponds to taking the sphere bundle of a vector bundle, and the fiber-homotopy geometric dimension of a stable spherical fibration is the smallest r such that the classifying map lifts to BSG_r. Since the map $V_r \to QP_r$ utilized in (2.2) factors through SG/SG_r, Remark 1.2 follows immediately from the fact that in the cases covered by 1.1(i) a map $\Sigma^{-1}P^n \to V_r$ is null-homotopic if and only if its composite into $\Omega^\infty(P_r \wedge bJ)$ is null-homotopic.

3. Resolutions. In this section we will prove some results about (unstable) Adams resolutions and stable bo-resolutions which will be used in §4 to prove the lifting part of Theorem 1.1.

DEFINITION 3.1. Let X be a space of finite type. A resolution of X is a diagram

$$\begin{array}{ccccccc} X & \stackrel{p_0}{\leftarrow} & X^{\langle 1\rangle} & \stackrel{p_1}{\leftarrow} & X^{\langle 2\rangle} & \leftarrow & \cdots \\ \downarrow i_0 & & \downarrow i_1 & & \downarrow i_2 & & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

where K_i is a product of Eilenberg-Mac Lane spaces $K(\mathbf{Z}_2, n)$ for $i > 0$, and K_0 is a product of $K(\mathbf{Z}_2, n)$'s and possibly also $K(\mathbf{Z}, n)$'s, and $\Omega K_{s-1} \to X^{\langle s \rangle} \to^{p_s} X^{\langle s-1 \rangle}$ is the fibration classified by i_{s-1}. If also $i_s^*\colon H^*(K_s) \to H^*(X^{\langle s \rangle})$ is surjective for all s, this is an Adams resolution. (Here and throughout the paper all cohomology groups have $\mathbf{Z}_2$-coefficients unless indicated otherwise.) An Adams resolution through dimension d is a resolution in which the $K(\mathbf{Z}_2, n)$'s have $n \leqslant d$ and i_s^* is surjective through dimension d. An Adams resolution of a spectrum is defined similarly. If X is a space, then applying $\Omega^\infty(\ )$ to an Adams resolution of the spectrum $\Sigma^\infty X$ yields a resolution

$$\begin{array}{ccccccc} QX & \leftarrow & \underline{X^{\langle 1 \rangle}} & \leftarrow & \underline{X^{\langle 2 \rangle}} & \leftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

which may not be an Adams resolution but satisfies $E_2^{s,t} \approx \operatorname{Ext}_{\mathcal{Q}}^{s,t}(H^*X, \mathbf{Z}_2)$.

If $\pi_*(\ )$ is applied to a resolution of X, a spectral sequence is obtained with $E_1^{s,t+s} = \pi_t(K_s)$ and $E_\infty^{*,t+*}$ an associated graded group to $\pi_t(X)$ if $X^{\langle \infty \rangle}$ is acyclic.

PROPOSITION 3.2. (i) *If $f\colon X \to Y$ is a map, $\{X^{\langle s \rangle}\}$ is an Adams resolution, and $\{Y^{\langle s \rangle}\}$ is a resolution, there exists a map of resolutions $\{X^{\langle s \rangle} \to Y^{\langle s \rangle}\}$.*

(ii) *If $\{X^{\langle s \rangle} \to Y^{\langle s \rangle}\}$ is a map of resolutions covering a map $f\colon X \to Y$, then there is a resolution of the fiber F of f such that there is an exact sequence*

$$\cdots \to E_2^{s,t}(F) \to E_2^{s,t}(X) \to E_2^{s,t}(Y) \to E_2^{s+1,t}(F) \to \cdots.$$

PROOF OF (ii). Let $F^{\langle s \rangle}$ be the fiber of $X^{\langle s \rangle} \to Y^{\langle s-1 \rangle}$. Then $F^{\langle s+1 \rangle} \to F^{\langle s \rangle}$ is classified by $K_s(X) \times \Omega K_{s-1}(Y)$, so that $0 \to E_1^{s-1,t}(Y) \to E_1^{s,t}(F) \to E_1^{s,t}(X) \to 0$ is exact. □

THEOREM 3.3. *Suppose $\{g_s\}$ is the map of resolutions induced by the map $V_r \to QP_r$ considered in §2. In the diagram*

$$\begin{array}{ccccc} & & V_r^{\langle s \rangle} & \xrightarrow{g_s} & \underline{P_r^{\langle s \rangle}} \\ & \nearrow^{l} & & & \\ \Sigma^{-1}P^n & & \downarrow & & \\ & \searrow_{f} & & & \\ & & V_r^{\langle s-1 \rangle} & & \end{array}$$

if $n \leqslant 2r + 2s - 4$ and $[g_s \circ l] = 0$, then $[f] = 0$.

PROOF. Let $X_{r+i,i}$ denote the fibre of $V_{r+i,i} \to Q(P_r^{r+i-1})$,

$$X_r = \text{fibre}(V_r \to Q(P_r)),$$

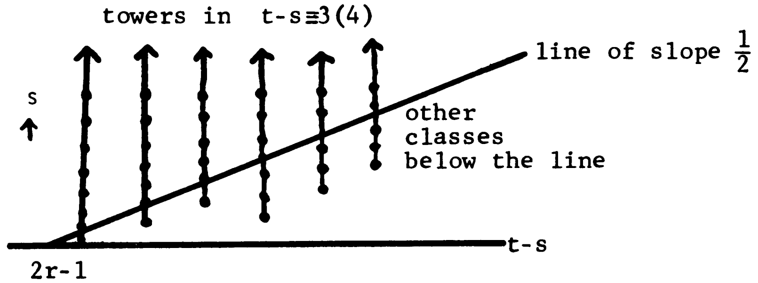
and $F_{r+i} = \text{fibre}(S^{r+i} \to Q(S^{r+i}))$. There is a diagram

$$\begin{array}{ccccc} X_{r+i,i} & \to & X_{r+i+1,i+1} & \to & F_{r+i} \\ \downarrow & & \downarrow & & \downarrow \\ V_{r+i,i} & \to & V_{r+i+1,i+1} & \to & S^{r+i} \\ \downarrow & & \downarrow & & \downarrow \\ Q(P_r^{r+i-1}) & \to & Q(P_r^{r+i}) & \to & Q(S^{r+i}) \end{array}$$

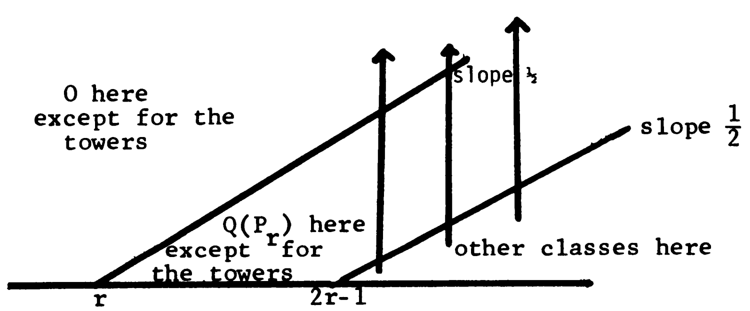
in which the rows and columns are fibrations. By 3.2(ii) resolutions can be chosen so that these fibrations induce long exact sequences in $E_2^{*,*}$. By [3], $E_2^{s,s+j}(F_m) = 0$ if $j < 2m + 2s - 2$, except for $E_2^{s,s+2m-1}(F_m) = \mathbf{Z}_2$ if m is even. Since $X_r = \varinjlim_i X_{r+i,i}$ and $X_{r+1,1} = F_r$, the exact sequences in $E_2^{*,*}$ obtained from the first row of the above diagram for all $i \geqslant 1$ show that $E_2(X_r)$ is built up from $E_2(F_m)$ for $m \geqslant r$. Hence by Curtis's vanishing theorem above, if $j < 2r + 2s - 2$,

$$E_2^{s,s+j}(X_r) = \begin{cases} \mathbf{Z}_2, & j \equiv 3\ (4), j \geqslant 2r-1, \\ 0, & \text{otherwise.} \end{cases}$$

Pictorially, the E_2-chart for X_r with the usual $(t - s, s)$ grading is



The exact sequence in E_2 induced by $X_r \to V_r \to Q(P_r)$ now implies that $E_2(V_r)$ has the following rough form:



If the composite $\Sigma^{-1}P^n \to^l V_r^{\langle s\rangle} \to^{g_s} P_r^{\langle s\rangle}$ is null-homotopic, then l factors through $\Sigma^{-1}P^n \to X_r^{\langle s\rangle} \to^i V_r^{\langle s\rangle}$. Through dimension $n - 1$,

$$X_r^{\langle s\rangle} = \prod_{j > 2r-1; j \equiv 3\ (4)} K(\mathbf{Z}, j),$$

and the map $X_r^{\langle s\rangle} \to X_r^{\langle s-1\rangle}$ is multiplication by 2 on each $K(\mathbf{Z}, j)$. Thus the composite $\Sigma^{-1}P^n \to X_r^{\langle s\rangle} \to X_r^{\langle s-1\rangle}$ is the image of

$$\bigoplus_{j > 2r; j \equiv 0\ (4)} H^j(P^n; \mathbf{Z}) \to^2 \bigoplus_{j > 2r; j \equiv 0\ (4)} H^j(P^n; \mathbf{Z}),$$

which is 0. □

REMARK 3.3.1. Theorem 3.3 can be strengthened slightly by using the fact that F_m is $\Sigma^{m-1}P_m$ through a range, so that in 3.3: if $\mathrm{Ext}_{\mathcal{Q}}^{s,t}(H^*(\Sigma^{r-1}P_r), \mathbf{Z}_2) = 0$ for $2r - 1 < t - s \leqslant n - 1$ and $[g_s \circ l] = 0$, then $[f] = 0$. This upper edge of Ext is known from [1] or [13].

THEOREM 3.4. *If* $f\colon \Sigma^{-1}P^{n}_{(n+1)/2} \to SO$ *classifies* $m\xi$, *then* f *lifts to* $SO^{\langle \nu(m)-[\log_2 n]\rangle}$.

PROOF. If $0 \leqslant b \leqslant 3$, let $\rho(4a+b) = 8a + 2^b$; thus $\rho(n)$ is the $(n+1)$st positive integer $\equiv 0, 1, 2$, or 4 (8). If X is any space or spectrum, let $X[j] \to X$ denote a map from a $(j-1)$-connected space which induces an isomorphism in $\pi_i(\ )$ for $i \geqslant j$. We shall need

LEMMA 3.5 [11]. $H^i(SO[\rho(k)-1]) \to H^i(SO[\rho(k+1)-1])$ *is* 0 *for* $i < 2^{k+1} - 1$.

We form the unstable Adams resolution of SO through dimension $n-1$. There is a commutative diagram (where $l = [\log_2 n]$)

$$\begin{array}{ccc}
SO[\rho(\nu(m))-1] & \overset{r_{\nu(m)-l}}{\longrightarrow} & SO^{\langle \nu(m)-l\rangle} \\
\downarrow q_{\nu(m)} & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
SO[\rho(l+1)-1] & \overset{r_1}{\longrightarrow} & SO^{\langle 1\rangle} \\
\downarrow q_{l+1} & & \downarrow p_0 \\
SO[\rho(l)-1] & \overset{q_1\circ\cdots\circ q_l}{\longrightarrow} & SO
\end{array}$$

which exists because, by 3.5, $H^i(q_j) = 0$ for $i \leqslant n-1$ and $j > l$, and each p_j is induced by a map into Eilenberg-Mac Lane spaces with generators in degree $\leqslant n-1$. Since $m\xi$ is trivial on $P^{\rho(\nu(m))-1}$, f lifts to $SO[\rho(\nu(m))-1]$. Following by $r_{\nu(m)-l}$ yields the result. □

If X is a space, the bo-resolution of X is the diagram of spectra

$$\begin{array}{ccccccc}
\Sigma^\infty X & \overset{q_0}{\longleftarrow} & X_1 & \overset{q_1}{\longleftarrow} & X_2 & \overset{q_2}{\longleftarrow} & \cdots \\
\downarrow j_0 & & \downarrow j_1 & & \downarrow j_2 & & \\
X\wedge bo & & X_1\wedge bo & & X_2 \wedge bo & &
\end{array}$$

where q_s is the fibration induced by j_s. Let I denote the fiber of the unit map $\mathbb{S} \to bo$, and $I^{\wedge s} = I \wedge \ldots \wedge I$ with s factors. Then $X_s = X \wedge I^{\wedge s}$. Splicing the homotopy exact sequences of these fibrations yields a cochain complex $C_i(X)$:

$$\pi_i(X\wedge bo) \overset{d_0}{\to} \pi_i(X\wedge \overline{bo}\wedge bo) \overset{d_1}{\to} \pi_i\left(X\wedge \overline{bo}^{\wedge 2}\wedge bo\right) \overset{d_2}{\to} \cdots$$

where $\overline{bo} = \Sigma I = \text{cofibre}(\mathbb{S} \to^i bo)$ and $d_i = (X\wedge \overline{bo}^{\wedge i}) \wedge (bo \to \overline{bo}) \wedge i$.

The main theorem of bo-resolutions, adapted to the space X, states

THEOREM 3.6. *Suppose* X *is* $(m-1)$-*connected and there are no nontrivial differentials in the Adams spectral sequence for* $X \wedge bo$ *through dimension* i. *Then the cohomology of the cochain complex* $C_i(X)$ *satisfies*

$$H^s(C_i(X)) = \begin{cases} 0, & i < 4s+m, \\ V_{s,i}, & s \geqslant 2, i \geqslant 4s+m, \\ Q_i \oplus V_{1,i}, & s = 1, i > 4+m, \end{cases}$$

where $V_{s,i}$ is a $\mathbf{Z}_2$-vector space all elements of which are detected in $\mathbf{Z}_2$-cohomology, i.e. they have Adams filtration 0, *and*

$$Q_i = \operatorname{coker}\left(\pi_i(X \wedge bo) \xrightarrow{\theta_*} \pi_i(X \wedge \Sigma^4 bsp)\right),$$

where θ_* is as in §2.

The significance of this result is that bo-resolutions have the bJ-primary obstructions at levels 0 and 1, and above this they work *almost* like a $K(\mathbf{Z}_2)$-resolution, but they have the advantage over $K(\mathbf{Z}_2)$-resolutions of increasing the connectivity faster. To be more precise, we have

COROLLARY 3.6.1. *Suppose there are no nontrivial differentials through dimension N in the ASS for $D_N Y \wedge X \wedge bo$ where $D_N Y$ denotes a stable N-dual of a finite complex Y. Let X_s denote the sth spectrum in the bo-resolution of X. If $s \geqslant 2$ and l: $Y \to X_s$ is cohomologically trivial, then there exists a map l': $Y \to X_s$ such that $q_{s-1}l = q_{s-1}l'$: $Y \to X_{s-1}$ and l' lifts to X_{s+1}. (Here and later $Y \to X$ means $\Sigma^\infty Y \to X$ when Y is a space and X is a spectrum.)*

PROOF. In the diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}X_{s-1} \wedge bo & & X_{s+1} & & & & \\
 & \searrow^{i} & \downarrow & & & & \\
Y & \xrightarrow{\ \ l\ \ } & X_s & \xrightarrow{j} & X_s \wedge bo & \xrightarrow{d_s} & \Sigma X_{s+1} \wedge bo \\
 & & \downarrow q_{s-1} & & & & \\
 & & X_{s-1} & & & &
\end{array}$$

one smash factor of $d_s \cdot j$ is $S \to bo \to \overline{bo}$ so $d_s \cdot j$ is trivial. By applying 3.6 to $C_N(D_N Y \wedge X)$ and using S-duality,

$$\frac{\ker\left((d_s)_*\colon [Y, X_s \wedge bo] \to [Y, \Sigma X_{s+1} \wedge bo]\right)}{\operatorname{im}(ji)_*\colon [Y, \Sigma^{-1}X_{s-1} \wedge bo] \to [Y, X_s \wedge bo]\big)}$$

is a $\mathbf{Z}_2$-vector space all elements of which are detected in $\mathbf{Z}_2$-cohomology. Since $[jl] \in \ker(d_{s*})$ and $H^*(jl) = 0$, there exists ϕ: $Y \to \Sigma^{-1}X_{s-1} \wedge bo$ such that $[ji\phi] = [jl]$. Then $l' = l - i\phi$ is the desired map. □

The remainder of this section will be devoted to the proof of 3.6. It follows quite directly from the methods of **[14]**, but we hope to clarify some aspects of that proof. No aspects of this proof will be required in §4.

Let W denote the fiber of $\Omega^2 S^3 \to S^1$. Then $\Omega^2 S^3 \simeq W \times S^1$. The May filtration on $\Omega^2 S^3$ induces a filtration $F_n(W)$. Let $\overline{B}(n)$ denote the Thom spectrum defined by the composite $F_{2n}(W) \hookrightarrow \Omega^2 S^3 \xrightarrow{\Omega^2 g} BO$. Let $B_n = \Sigma^{4n}\overline{B}(n)$ and if $\bar{n} = (n_1, \ldots, n_s)$ let $B_{\bar{n}} = B_{n_1} \wedge \cdots \wedge B_{n_s}$. Let R_s denote the set of s-tuples of positive integers. If $(n_1, \ldots, n_s) \in R_s$ and $\varepsilon = 0$ or 1, we say that $(n_1, \ldots, n_{i+1}, k, n_i - k, n_{i+1}, \ldots, n_s)$ is an *ε-successor* of $(n_1, \ldots, n_s)$ if $\nu\binom{n_i}{k} = \varepsilon$.

A map of spectra $f\colon X \to Y$ is a *homotopy equivalence* mod $K(\mathbf{Z}_2)$'s if there are wedges of $K(\mathbf{Z}_2)$'s, K_1 and K_2, and homotopy equivalences h_1 and h_2 so that the $X_1 \to Y_1$ component of

$$X_1 \vee K_1 \underset{h_1}{\overset{\simeq}{\to}} X \underset{f}{\to} Y \underset{h_2}{\overset{\simeq}{\to}} Y_1 \vee K_2$$

is a homotopy equivalence. The map of spaces $\Omega^\infty X \to \Omega^\infty Y$ is also called a homotopy equivalence mod $K(\mathbf{Z}_2)$'s.

The bulk of the work in proving 3.6 is in

THEOREM 3.7. *There are homotopy equivalences*

$$h_s\colon \bigvee_{\bar n \in R_s} B_{\bar n} \wedge bo \to \overline{bo}^{\wedge s} \wedge bo$$

such that, for the map d_s *which induces the chain complexes* $C(S^0)$ *of* 3.6

$$\bigvee_{\bar n \in R_s} B_{\bar n} \wedge bo \overset{h_s}{\to} \overline{bo}^{\wedge s} \wedge bo \overset{d_s}{\to} \overline{bo}^{\wedge (s+1)} \wedge bo \overset{h_{s+1}^{-1}}{\to} \bigvee_{\bar m \in R_{s+1}} B_{\bar m} \wedge bo,$$

the component $B_{\bar n} \wedge bo \to B_{\bar m} \wedge bo$

(i) *is a homotopy equivalence* mod $K(\mathbf{Z}_2)$*'s if* $\bar m$ *is a* 0*-successor of* $\bar n$,

(ii) *lifts to a homotopy equivalence* mod $K(\mathbf{Z}_2)$*'s* $B_{\bar n} \wedge bo \to (B_{\bar m} \wedge bo)^{\langle 1 \rangle}$ *if* $\bar m$ *is a* 1*-successor of* $\bar n$,

(iii) *has filtration* > 1 *if* $|\bar m| < |\bar n|$, *or* $|\bar m| = |\bar n|$ *and* $\bar m$ *is neither a* 0- *nor a* 1*-successor of* $\bar n$. (*Here and elsewhere* $|\bar n| = \sum n_i$.)

We shall return to the proof of 3.7 at the end of this section. 3.7 is easily adapted to spaces X satisfying the hypotheses of 3.6.

COROLLARY 3.8. *Suppose the Adams spectral sequences*

$$\operatorname{Ext}_{\mathcal{Q}_1}(H^*X, \mathbf{Z}_2) \Rightarrow \pi_*(X \wedge bo)$$

has no nontrivial differentials. If $X \wedge$ *is applied to the maps of* 3.7, *the component* $X \wedge B_{\bar n} \wedge bo \to X \wedge B_{\bar m} \wedge bo$

(i) *is a homotopy equivalence* mod $K(\mathbf{Z}_2)$*'s if* $\bar m$ *is a* 0*-successor of* $\bar m$,

(ii) *lifts to a map* $X \wedge B_{\bar n} \wedge bo \to (X \wedge B_{\bar m} \wedge bo)^{\langle 1 \rangle}$ *which is surjective in* $\pi_*(\ )$ *and whose kernel consists at most of filtration zero* $\mathbf{Z}_2$*'s, if* $\bar m$ *is a* 1*-successor of* $\bar n$,

(iii) *has filtration* > 1 *if* $|\bar m| < |\bar n|$, *or* $|\bar m| = |\bar n|$ *and* $\bar m$ *is neither a* 0- *nor* 1*-successor of* $\bar n$.

If there are no differentials through dimension i, *then the results are true through dimension* i.

PROOF. (i) is clear, and (iii) follows immediately from 3.7(iii) and the map $X \wedge (B_{\bar m} \wedge bo)^{\langle 2 \rangle} \to (X \wedge B_{\bar m} \wedge bo)^{\langle 2 \rangle}$. To prove (ii) we use 3.7(ii) to obtain

$$X \wedge B_{\bar n} \wedge bo \overset{\simeq}{\to} X \wedge (B_{\bar m} \wedge bo)^{\langle 1 \rangle} \overset{l}{\to} (X \wedge B_{\bar m} \wedge bo)^{\langle 1 \rangle}.$$

A standard result in homological algebra (e.g. [17, Exercise 3, p. 102]) shows that applying $\operatorname{Ext}^s_{\mathcal{Q}}(H^*(\ ), \mathbf{Z}_2)$ to l induces an isomorphism when $s > 0$ and an epimorphism when $s = 0$. We will be done once we note that the ASS for $X \wedge B_{\bar n} \wedge bo$

has no nontrivial differentials. This follows from the following lemma which will be proved later in this section.

LEMMA 3.9. *If* $\bar{n} = (n_1, \ldots, n_s)$, *let* $|\bar{n}| = \Sigma\, n_i$ *and* $\alpha(\bar{n}) = \Sigma\, \alpha(n_i)$.

$$B_{\bar{n}} \wedge bo \simeq K \vee \begin{cases} \Sigma^{4|\bar{n}|} bo^{\langle 2|\bar{n}| - \alpha(\bar{n})\rangle} & \text{if } |\bar{n}| \text{ is even,} \\ \Sigma^{4|n|} bsp^{\langle 2|\bar{n}| - 1 - \alpha(\bar{n})\rangle} & \text{if } |\bar{n}| \text{ is odd,} \end{cases}$$

where K *is a wedge of suspensions of* $K(\mathbf{Z}_2)$.

Since there is a map $X \wedge (bo^{\langle t\rangle}) \to (X \wedge bo)^{\langle t\rangle}$ inducing an isomorphism in $\mathrm{Ext}^s(\ )$ for $s > 0$, $X \wedge B_{\bar{n}} \wedge bo$ has no nontrivial differentials when $|\bar{n}|$ is even. A similar argument works when $|\bar{n}|$ is odd once we note that there are no differentials in the ASS for $X \wedge bsp$. This follows from the fact that there are maps $\Sigma^8 bsp \to \Sigma^4 bo \to bsp$ such that the composite induces an injection in $\mathrm{Ext}_{\mathcal{Q}}(H^*(X \wedge\ ), \mathbf{Z}_2)$. □

In order to see that 3.8 implies 3.6, we shall use the following combinatorial result, which orders the summands in $\bigvee_{\bar{n} \in R_s} B_{\bar{n}} \wedge bo$. It may be viewed as an elaboration upon the argument used in [**14**, proof of 5.11].

LEMMA 3.10. *There are partitions* $R_s = T_s \amalg U_s$, *total orderings* $\ll$ *on* T_s *and* U_s, *and bijections* Φ: $T_s \to U_{s+1}$ *so that* $\Phi(\bar{n})$ *is the first* 0-*successor of* $\bar{n}$ *if* $\bar{n}$ *contains some non*-2-*powers, and* $\phi(\bar{n})$ *is the first* 1-*successor of* $\bar{n}$ *if* $\bar{n}$ *consists of all* 2-*powers. Also*, $|\bar{m}| < |\bar{n}|$ *implies* $\bar{m} \ll \bar{n}$.

We shall return to the rather detailed proof of 3.10 after using it to deduce 3.6.

PROOF OF 3.6. The case $s = 1$ follows from the methods below together with those of [**14**] and [**16**], so we shall assume $s \geqslant 2$. The first case of 3.6 follows immediately from the fact that bo is 3-connected.

We may ignore the split $K(\mathbf{Z}_2)$'s in the equivalences mod $K(\mathbf{Z}_2)$'s of 3.8. We choose homotopy classes g_α corresponding to as basis of $\mathrm{Ext}^{0,*}_{\mathcal{Q}}(H^*(B_{\Phi(\bar{n})} \wedge bo))$ for all $\bar{n} \in T_{s-1}$ which consist only of 2-powers. We first show that any element y of $\pi_*(X \wedge \bar{bo}^{\wedge s} \wedge bo)$ plus perhaps some g_α's is equivalent mod $\mathrm{im}(d_{s-1})$ to an element of $\pi_*(X \wedge \bigvee_{\bar{n} \in T_s} B_{\bar{n}} \wedge bo)$. It suffices to show that this is true up to any filtration.

If $z \in \pi_*(X \wedge \bar{bo}^{\wedge s} \wedge bo)$, let $z_{\bar{n}}$ denote the component of z in $\pi_*(X \wedge B_{\bar{n}} \wedge bo)$. Suppose we have shown that for $\hat{y} = y + d_{s-1}(w) + \Sigma\, g_\alpha$, $\mathrm{filtr}(\hat{y}_{\bar{n}}) \geqslant j$ for all $\bar{n} \in U_s$, $\mathrm{filtr}(\hat{y}_{\bar{n}}) > j$ for all $\bar{n} \ll \bar{l}$, and $\mathrm{filtr}(\hat{y}_{\bar{l}}) = j$. If $j = 0$ and $\bar{l} = \phi(\bar{e})$ where $\bar{e}$ has all 2-powers, then by adding an appropriate $g_\gamma \in \pi_*(X \wedge B_{\bar{l}} \wedge bo)$, we can make $\mathrm{filtr}(\hat{y} + g_\gamma)_{\bar{l}} \geqslant 1$. Otherwise, by 3.10 and 3.8 there exists $v \in \pi_*(X \wedge B_{\bar{e}} \wedge bo)$ where $l = \phi(\bar{e})$, such that $(d_{s-1}v)_{\bar{l}} = \hat{y}_{\bar{l}}$. Moreover, we still have

$$\mathrm{filtr}(\hat{y} - d_{s-1}v)_{\bar{n}} > j \quad \text{for all } \bar{n} \ll \bar{l}$$

and

$$\mathrm{filtr}(\hat{y} - d_{s-1}v)_{\bar{n}} \geqslant j \quad \text{for all } \bar{n} \in U_s.$$

This inductive procedure enables us to eliminate all U_s-components.

We will now show that d_s is injective on $\pi_*(X \wedge \bigvee_{\bar{n}\in T_s} B_{\bar{n}} \wedge bo)$; by the preceding paragraphs this will imply 3.6. We partition $T_s = P_s \amalg Q_s$, where P_s are those tuples consisting of all 2-powers. If $y \neq 0 \in T_s$, let

$$j = \min(\{\mathrm{filtr}(y_\alpha)\colon \alpha \in Q_s\} \cup \{\mathrm{filtr}(y_\alpha) + 1\colon \alpha \in P_s\})$$

and let α_0 be the smallest α realizing j. By 3.10 and 3.8 the $\phi(\alpha_0)$-component of $d_s y$ is nonzero. □

PROOF OF 3.10. Let $T_s = \{(2^{e_1}, \dots, 2^{e_s})\colon e_1 > e_2 \geqslant e_3 \geqslant \cdots \geqslant e_s\} \cup \{(n_1, \dots, n_{j-1}, 2^{e_j} + k, 2^{e_{j+1}}, \dots, 2^{e_s})\colon 0 < k < 2^{e_j},\ e_j \geqslant e_{j+1} \geqslant \cdots \geqslant e_s\}$, $\phi(2^{e_1}, \dots, 2^{e_s}) = (2^{e_1-1}, 2^{e_1-1}, 2^{e_2}, \dots, 2^{e_s})$, and

$$\phi(n_1, \dots, n_{j-1}, 2^{e_j} + k, 2^{e_{j+1}}, \dots, 2^{e_s}) = (n_1, \dots, n_{j-1}, k, 2^{e_j}, 2^{e_{j+1}}, \dots, 2^{e_s}).$$

ϕ is easily checked to be a bijection $T_s \to R_{s+1} - T_{s+1}$ which preserves $|\ |$.

We shall call the elements of T_s which contain all 2-powers "type B" and the other elements "type A". A *successor* is a 0-successor of a type A element or a 1-successor of a type B element. Note that $\Phi(\bar{n})$ is always a successor of $\bar{n}$. Define a relation $\ll$ on T_s by

(∗) if $\bar{n} \neq \bar{m} \in T_s$ and $\phi(\bar{n})$ is a successor of $\bar{m}$, then $\bar{m} \ll \bar{n}$.

We now show that (∗) causes no loops $\bar{m} \ll \bar{n}_1 \ll \dots \ll \bar{n}_{q-1} \ll \bar{m}$. Suppose such a loop exists.

Case 1. There are no type B's in the sequence. If $e_1 < \dots < e_r$, let $p(\sum_{i=1}^r 2^{e_i}) = (2^{e_1}, \dots, 2^{e_r})$ and let $p(a_1, \dots, a_s) = (p(a_1), \dots, p(a_s))$. Then $p(\bar{m}) < p(\bar{n}_1) < \dots < p(\bar{n}_{q-1}) < p(\bar{m})$ under the lexicographical ordering from the left. But this cannot happen since the lexicographic order is a total order.

Case 2. There is at least one type B. By considering the various type A or B possibilities, one shows that $\bar{m} \ll \bar{n}$ under (∗) implies ν(first component of $\bar{m}$) $\leqslant$ ν(first component of $\bar{n}$) and if $\bar{n}$ has type B then this inequality is strict. This implies that there can be no looping under (∗) when a type B occurs.

Since there are no loops, $\ll$ can be extended to a total order on T_s (compatible with $|\ |$). If U_{s+1} is ordered by $\phi(\ll)$, then (∗) guarantees that $\phi(\bar{m})$ is the first successor of $\bar{m}$. □

PROOF OF 3.9. Except for the construction of maps near the end, this argument was essentially given in [14], but again some clarification seems worthwhile.

Let $\mathrm{filtr}(\xi_1^{i_1}\xi_2^{i_2}\dots) = \sum i_j 2^{j-1}$. Then $\chi(H^*\bar{B}(n))^*$ is the subspace S_n of $\mathbf{Z}_2[\xi_1^2, \xi_2, \dots]$ spanned by monomials of filtration $\leqslant 2n$. This is proved by noting that it is $\ker(R_{\mathrm{Sq}^1}, R_{\mathrm{Sq}^{2n+2}}, \dots\colon \mathcal{A}^* \to \bigoplus \mathcal{A}^*)$.

For $j \in \mathbf{Z}_4$ and $n \geqslant 0$, we define $\mathcal{A}_1$-modules $Q_{j,n}$ by induction on n by the nontrivial extension of $\mathcal{A}_1$-modules

$$0 \to \{1, \mathrm{Sq}^2, \mathrm{Sq}^3, \mathrm{Sq}^2\mathrm{Sq}^3\} \to Q_{j,n} \to \Sigma^4 Q_{j,n-1} \to 0$$

with $Q_{0,0} = \mathbf{Z}_2$, $Q_{1,0} = \{1, \mathrm{Sq}^2, \mathrm{Sq}^3\}$, $Q_{2,0} = \Sigma^{-2}\{1, \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^2\mathrm{Sq}^1, \mathrm{Sq}^3\mathrm{Sq}^1\}$, $Q_{3,0} = \Sigma^{-3}\{1, \mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^1\}$. Note that $Q_{j,n}$ is P_{ijn}/N_i in the notation of [4, 3.6] (with $j = 1$ and 3 reversed) and are the modules T, S, Y, and Z of [18, pp. 133–135]. The next lemma is a restatement of [18, 1.15]. It can also be proved by the method of [4, 3.9].

LEMMA 3.11. *There is an isomorphism of $\mathcal{A}_1$-modules*

$$Q_{j_1,n_1} \otimes Q_{j_2,n_2} \approx F \oplus Q_{j_1+j_2,n_1+n_2+\delta}$$

where F is a free $\mathcal{A}_1$-module and

$$\delta = \begin{cases} -1 & \text{if } \{3\} \subset \{j_1, j_2\} \subset \{2, 3\}, \\ 1 & \text{if } \{1\} \subset \{j_1, j_2\} \subset \{1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.12. *There is an isomorphism of $\mathcal{A}_1$-modules*

$$H^*(\bar{B}(n_1) \wedge \cdots \wedge \bar{B}(n_s)) \approx F \oplus Q_{\Sigma\alpha(n_j), \Sigma n_j - D}$$

where $D = 2l$ if $\Sigma\, \alpha(n_j) = 4l$, and $D = 2l + 1$ if $4l + 1 \leq \Sigma\, \alpha(n_j) \leq 4l + 3$, where F is a free $\mathcal{A}_1$-module.

PROOF. The $\mathcal{A}_1$-submodule T_{2^i} of S_{2^i} generated over $\mathcal{A}_1$ by ξ_{i+2} and $\xi_j^{8(2^{i-1-j}-1-k)}\xi_{j+1}^{4k+2}\xi_{j+2}$, $0 \leq k \leq 2^{i-1-j} - 1$, $0 \leq j \leq i - 1$, has its dual isomorphic to $Q_{1,2^i-1}$. Multiplying out $\bigotimes_{2^i \in n} T_{2^i} \to S_n$ induces an isomorphism in Q_0- and Q_1-homology. The result now follows from several applications of 3.11. □

The following result is our restatement of [**18**, 2.4].

LEMMA 3.13. *There are isomorphisms of $\mathcal{A}$-modules*

$$H^*(bo^{\langle n\rangle}) \approx \mathcal{A} \otimes_{\mathcal{A}_1} Q_{-n,(n+\varepsilon_1)/2},$$

$$H^*(bsp^{\langle n\rangle}) \approx \mathcal{A} \otimes_{\mathcal{A}_1} Q_{1-n,(n+\varepsilon_2)/2}$$

where

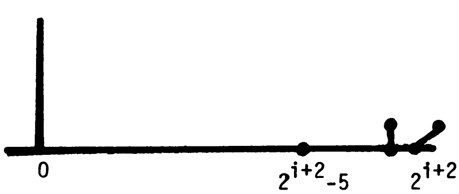
$$\varepsilon_1 = \begin{cases} 1, & n \equiv 1\ (4), \\ -1, & n \equiv 3\ (4), \\ 0, & n \text{ even}, \end{cases}$$

and

$$\varepsilon_2 = \begin{cases} 2, & n \equiv 2\ (4), \\ 0, & n \equiv 0\ (4), \\ 1, & n \text{ odd}. \end{cases}$$

3.12 and 3.13 imply that the spaces in 3.9 have isomorphic cohomology modules. It remains to construct the maps realizing the isomorphisms. The main step is the map $\bar{B}(2^i) \to bo^{\langle 2^{i+1}-1\rangle}$.

By the Adams edge theorem [**1**], a chart for the ASS of $\mathbb{S}^{\langle 2^{i+1}-1\rangle}$ begins



Thus the inclusion of the bottom cell of $\bar{B}(2^i)$ into $\mathbb{S}^{\langle 2^{i+1}-1\rangle}$ extends over the $(2^{i+2}-5)$-skeleton. The attaching map for the next cell has positive filtration in $\pi_{2^{i+2}-5}(\bar{B}(2^i)^{(2^{i+2}-5)})$ and hence maps to 0 in $\pi_{2^{i+2}-5}(\mathbb{S}^{\langle 2^{i+1}-1\rangle})$. Since $H^j(\bar{B}(2^i)) = 0$ for $j \geq 2^{i+2}$, there are no obstructions to extending the map to $\bar{B}(2^i) \to \mathbb{S}^{\langle 2^{i+1}-1\rangle}$. Following into $bo^{\langle 2^{i+1}-1\rangle}$ yields our desired map.

After applying $\wedge bo$, the map $\bigwedge_{2^i \in n} \bar{B}(2^i) \to \bar{B}(n)$ discussed in **[12]** and **[14]** becomes a homotopy equivalence mod $K(\mathbf{Z}_2)$'s. (This follows from 3.12 or from **[14]**.) A homotopy inverse on the irreducible part followed by the smash product of the maps constructed above and the map $\bigwedge_{2^i\in n} bo^{\langle 2^{i+1}-1\rangle} \to bo^{\langle 2n-\alpha(n)\rangle}$ yields the desired map $\bar{B}(n) \to bo^{\langle 2n-\alpha(n)\rangle}$ if n is even. The case n odd and the case of products of $\bar{B}(n)$'s follow from these methods using also the maps $bsp \wedge bo \to bsp$ and $bsp \wedge bsp \to bo^{\langle 2\rangle}$. $\square$

We now proceed toward the proof of 3.7, the final piece of unfinished business.

LEMMA 3.14. *There is an isomorphism of $\mathcal{Q}_1$-modules*

$$\bigoplus_{n\geq 0} \chi H_* B_n \xrightarrow{\oplus \chi f_{n*}} \chi H_* bo$$

given by $\chi f_{n*}(\xi_1^{i_1}\xi_2^{i_2}\dots) = \xi_1^{4n-\Sigma 2^j i_j}\xi_2^{i_1}\xi_3^{i_2}\dots.$

PROOF. By [2], $\chi H_* bo = \mathbf{Z}_2[\xi_1^4, \xi_2^2, \xi_3, \dots]$. χf_{n*} is an $\mathcal{Q}_1$-morphism, injective, and onto the filtration $4n$ part of $\chi H_* bo$. $\square$

LEMMA 3.15. *Let* $m = (\mu \wedge \mu) \circ (1 \wedge T \wedge 1)$: $(bo \wedge bo) \wedge (bo \wedge bo) \to bo \wedge bo$. *Let* $\Phi_{n,m}$: $B_n \wedge B_m \to B_{n+m}$ *denote the multiplication. There are maps* f_n: $B_n \to bo \wedge bo$ *such that* $f_{n*}(i_n) = \xi_1^{4n} \otimes 1$, *and if* $n = 2^i + k$, $1 < k \leq 2^i$, *then* $f_n \circ \phi_{2^i,k} - m \circ (f_{2^i} \wedge f_k)$ *has filtration* ≥ 4.

PROOF. The maps will be constructed inductively to have the multiplicative property. The homology property will follow automatically. Indeed we will have $\chi f_{n*}(\xi_1^{i_1}\xi_2^{i_2}\dots) = \xi_1^{4n-\Sigma 2^j i_j}\xi_2^{i_1}\xi_3^{i_2}\dots \otimes 1$.

If $k < 2^i$ the element of filtration ≥ 4 is not required in the multiplicative property. We show that

$$[m(f_{2^i} \wedge f_k)] \in \operatorname{im}(\phi_{2^i,k}^*: [B_{2^i+k}, bo\wedge bo] \to [B_{2^i} \wedge B_k, bo \wedge bo]).$$

These groups are calculated from the ASS where E_2-terms are

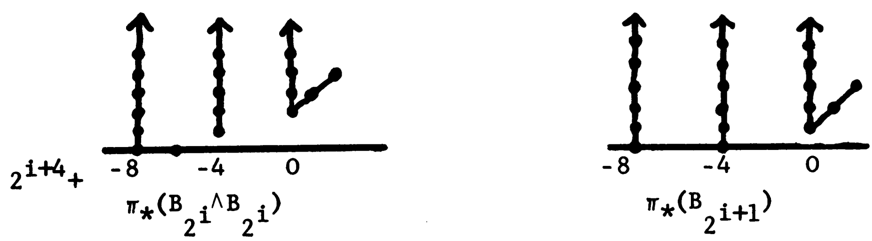
$$\operatorname{Ext}_{\mathcal{Q}_1}^{s,*}(\mathcal{Q}//\mathcal{Q}_1, H^*B_{2^i+k}) \to \operatorname{Ext}_{\mathcal{Q}_1}^{s,*}(\mathcal{Q}//\mathcal{Q}_1, H^*(B_{2^i}\wedge B_k)).$$

By 3.12 these homomorphisms fit into a long exact sequence with $\operatorname{Ext}_{\mathcal{Q}_1}(\mathcal{Q}//\mathcal{Q}_1$, free $\mathcal{Q}_1$-module), and hence are isomorphisms for $s > 0$. Thus it suffices to show that the cohomology homomorphism $(m(f_{2^i} \wedge f_k))^* \in \operatorname{im}(\phi_{2^i,k}^*)$, or equivalently $\chi(m(f_{2^i}\wedge f_k))_*(K) = 0$, where $K = \ker(\chi(\phi_{2^i,k})_*)$. But K is a free $\mathcal{Q}_1$-module spanned over $\mathbf{Z}_2$ by $\xi^{I_1}\otimes\xi^{J_1} + \xi^{I_2}\otimes\xi^{J_2}$ with $I_1 + J_1 = I_2 + J_2$, which is annihilated by the multiplication map $\chi(\phi_{2^i,k})_*$.

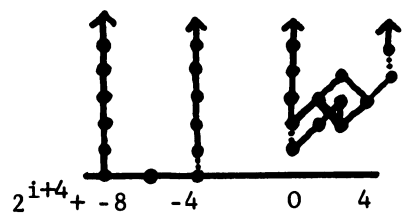
For the case $k = 2^i$, we will need the following.

PROPOSITION 3.16. *Let* $F_i = \Sigma^{2^{i+4}-5}M_2 \wedge \bar{B}(1)$. *There is a map* $F_i \xrightarrow{j} B_{2^i} \wedge B_{2^i}$ *whose cofiber is equivalent* mod $K(\mathbf{Z}_2)$*'s to* $B_{2^{i+1}}$.

PROOF. By 3.12, $\phi_{2^i,2^i}$ is a $(2^{i+4}-5)$-equivalence, and the homotopy charts are



There are no obstructions to extending the map $\Sigma^{2^{i+4}-5}M \to^p S^{2^{i+4}-4} \to^g B_{2^i} \wedge B_{2^i}$ over F_i. The composite $F_i \to^j B_{2^i} \wedge B_{2^i} \to B_{2^{i+1}}$ is trivial, so that there exists $MC(j) \to B_{2^{i+1}}$, which is easily seen to induce an isomorphism above filtration zero. Indeed the chart for $\pi_*(MC(j))$ is obtained from those of $B_{2^i} \wedge B_{2^i}$ and F as below:

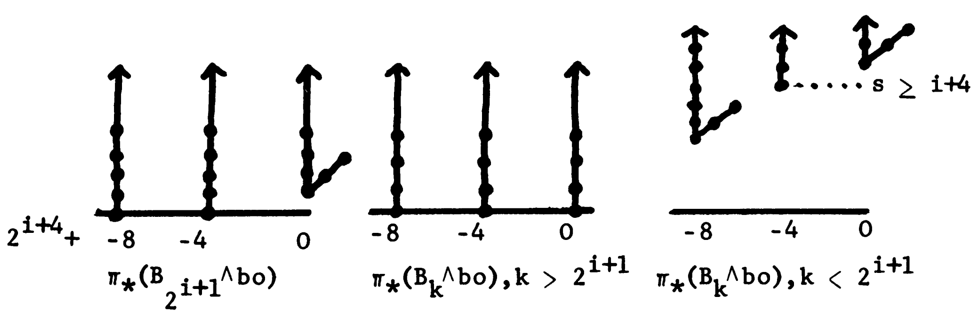


□

3.15 now follows immediately from the following calculation.

LEMMA 3.17. *If $m \in [B_{2^i} \wedge B_{2^i}, bo \wedge bo]$ satisfies $m_*(i_{2^i} \otimes i_{2^i}) = \xi_1^{2^{i+3}}$ there exists q of filtration $\geqslant i+4$ such that $j^*(m+q) = 0 \in [F_i, bo \wedge bo]$.*

PROOF. From 3.14 and 3.9, $\pi_*(bo \wedge bo)$ in the range $2^{i+4} - 8 < * < 2^{i+4} + 2$ is a sum of charts of the following three types:



Above filtration zero $[F_i, bo \wedge bo] \approx \pi_{2^{i+4}-4}(bo \wedge bo) \otimes \mathbf{Z}_2$. Since j^*m has positive filtration, it corresponds to some of the $\mathbf{Z}_2$'s in filtration $> i+4$ (or else is zero). Since we may assume by induction that $bo \wedge bo$ has been split through a range into $\bigvee B_k \wedge bo$ (or by just looking at the splitting of Ext-groups), it suffices to show:

if $k < 2^{i+1}$ and the generator of the $\mathbf{Z}$ in $\pi_{2^{i+4}-4}(B_k \wedge bo)$ has filtration l, then $[B_{2^i} \wedge B_{2^i}, B_k \wedge bo]$ contains a filtration l class f such that $j^*f \neq 0 \in [F_i, B_k \wedge bo]$.

This may be verified by direct calculation of Ext-groups or (as in [14]) by letting f be the restriction of the usual map

$$\begin{array}{ccc} \Sigma^{2^{i+3}} bo\langle 2^{i+2}-2\rangle & \to & \Sigma^{2^{i+3}} bo \\ \| & & \| \\ \Sigma^{2^{i+3}} B_{2^i} \wedge B_{2^i} \wedge bo & & (B_k \wedge bo)[2^{i+3}], \end{array}$$

where the first equality is by 3.9 and the second is easily proved by starting with the map $S^{2^{i+3}} \to B_k \wedge bo$ corresponding to a homotopy generator. □

Now we can prove the next result, which is a major step toward 3.7. Let $\Omega S^5_+ \wedge X_5 \to^h X_5 \wedge X_5$ be the homotopy equivalence constructed in [15]. Let o: $X_5 \to bo$ be the rational equivalence given by the bo-orientation. Let g: $\bigvee S^{4i} \to \Omega S^5$ be a fixed stable equivalence.

THEOREM 3.18. *There is a homotopy equivalence f such that the diagram*

$$\begin{array}{ccc} \bigvee S^{4n} \wedge X_5 & \xrightarrow{g\wedge 1} & \Omega S^5_+ \wedge X_5 \\ j\downarrow \wedge o & & \downarrow (o\wedge o)h \\ \bigvee B_n \wedge bo & \xrightarrow{f} & bo \wedge bo \end{array}$$

commutes modulo elements of filtration $\geqslant 4$, *where j is the inclusion of the bottom cell.*

PROOF. Suppose $n = 2^i + k$, $0 < k \leqslant 2^i$. In the diagram

$$\begin{array}{ccccc} S^{4\cdot 2^i} \wedge S^{4k} & \to & (\Omega S^5_+ \wedge X_5) \wedge (\Omega S^5_+ \wedge X_5) & \xrightarrow{(\mu\wedge\mu)\circ(1\wedge T\wedge 1)} & \Omega S^5_+ \wedge X_5 \\ \downarrow j & & (o\wedge o)h \downarrow (o\wedge o)h & & \downarrow (o\wedge o)h \\ B_{2^i} \wedge B_k & \xrightarrow{f_{2^i}\wedge f_k} & (bo\wedge bo)\wedge(bo\wedge bo) & \xrightarrow{(\mu\wedge\mu)\circ(1\wedge T\wedge 1)} & bo\wedge bo \end{array}$$

where the unlabeled arrow ulitizes g and the inclusion of the bottom cell of X_5, and f_{2^i} is the map constructed in 3.15, the right-hand square does not commute, but does when restricted to the bottom cell of X_5. (Look at the map of base spaces underlying h.) The left-hand square commutes up to elements of filtration $\geqslant 4$ by induction, and hence by the construction of f_n, we deduce that

$$\begin{array}{ccc} S^{4n} & \xrightarrow{i} & \Omega S^5_+ \wedge X_5 \\ j\downarrow & & \downarrow \\ B_n & \xrightarrow{f_n} & bo\wedge bo \end{array}$$

commutes up to elements of filtration 4.

Hence in the diagram

$$\begin{array}{ccccc} S^{4n} \wedge X_5 & \xrightarrow{i\wedge 1} & \Omega S^5_+ \wedge X_5 \wedge X_5 & \xrightarrow{1\wedge\mu} & \Omega S^5_+ \wedge X_5 \\ j\wedge o\downarrow & & \downarrow (o\wedge o)h)\wedge o & & \downarrow (o\wedge o)h \\ B_n \wedge bo & \xrightarrow{f_n\wedge 1} & bo\wedge bo\wedge bo & \xrightarrow{1\wedge\mu} & bo\wedge bo \end{array}$$

the left square commutes up to elements of filtration > 4, and the right square commutes (because the map $\Omega S^5 \times \Omega S^5 \to^q \Omega S^5 \times \Omega S^5$ underlying h^{-1} sends $(x, y) \to (x, x^{-1}y)$ thus satisfying $(1 \times m)(q \times 1) = q(1 \times m)$).

$f = \bigvee (1 \wedge \mu)(f_n \wedge 1)$ is a homotopy equivalence by 3.14. □

The homotopy equivalences of 3.7 are obtained by iterating 3.18. For example, h_2 is obtained as

$$(\bigvee B_n) \wedge (\bigvee B_n) \wedge bo \overset{1\wedge f}{\to} (\bigvee B_n) \wedge bo \wedge bo \overset{f\wedge 1}{\to} bo \wedge bo \wedge bo \overset{1\wedge \mu}{\to} bo \wedge bo.$$

It is easy to deduce from 3.18 that

$$\begin{array}{ccccc} \bigvee S^{4\bar{n}} \wedge X_5 & = & \bigvee S^{4n_1} \wedge \cdots \wedge \bigvee S^{4n_s} \wedge X_5 & \to & (\Omega S^5_+)^{\wedge s} \wedge X_5 \\ & & j_1 \wedge \cdots \wedge j_s \downarrow \wedge o & & \downarrow \\ \bigvee B_{\bar{n}} \wedge bo & = & \bigvee B_{n_1} \wedge \cdots \wedge \bigvee B_{n_s} \wedge bo & \to & (bo)^{\wedge s} \wedge bo \end{array}$$

commutes modulo elements of filtration ≥ 4.

We shall use X_5 to deduce information about the map d_s of 3.7 by the diagram

$$\begin{array}{ccccc} \to & (\Omega S^5)^{\wedge s} \wedge X_5 & \to & (\Omega S^5)^{\wedge (s+1)} \wedge X_5 & \to \\ & \downarrow & & \downarrow & \\ \to & (\bar{X}_5)^{\wedge s} \wedge X_5 & \overset{1\wedge p \wedge i}{\to} & (\bar{X}_5)^{\wedge (s+1)} \wedge X_5 & \to \\ & \downarrow & & \downarrow & \\ \to & \overline{bo}^{\wedge s} \wedge bo & \overset{d_s}{\to} & \overline{bo}^{\wedge (s+1)} \wedge bo & \to \\ & h_s \uparrow \simeq & & h_{s+1} \uparrow \simeq & \\ & \bigvee_{\bar{n} \in R_s} B_{\bar{n}} \wedge bo & & \bigvee_{\bar{m} \in R_{s+1}} B_{\bar{m}} \wedge bo & \end{array}$$

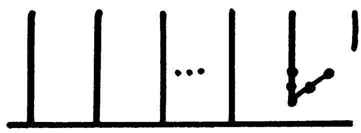
where the first row is the alternating sum of the diagonals. The first square commutes by [15, 3.8] or [14, 5.8]. By the remarks following 3.18 the diagram is equivalent to

$$\begin{array}{ccc} \bigvee_{\bar{n} \in R_s} S^{\bar{n}} \wedge X_5 & \to & \bigvee_{\bar{m} \in R_{s+1}} S^{\bar{m}} \wedge X_5 \\ \downarrow j \wedge o & & \downarrow j \wedge o \\ \bigvee_{\bar{n} \in R_s} B_{\bar{n}} \wedge bo & \to & \bigvee_{\bar{m} \in R_{s+1}} B_{\bar{m}} \wedge bo \end{array}$$

where the map in homology induced by the top map is

$$x^{i_1} \otimes \cdots \otimes x^{i_{s+1}} \mapsto \sum_j \sum_a (-1)^j \binom{i_j}{a} x^{i_1} \otimes \cdots \otimes x^{i_{j-1}} \otimes x^a \otimes x^{i_j - a} \otimes \cdots \otimes x^{i_{s+1}}.$$

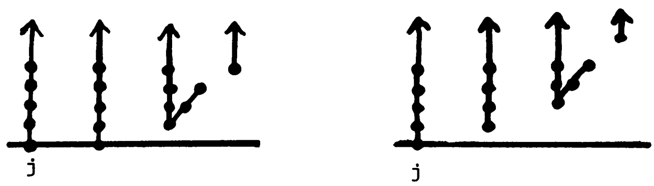
Proof of 3.7(i). If $\bar{m}$ is a 0-successor of $\bar{n}$, the map $S^{\bar{n}} \wedge X_5 \to S^{\bar{m}} \wedge X_5$ is a 2-equivalence. The map of Q_0-homology induced by $(j \wedge o)^*$: $H^*(B_{\bar{n}} \wedge bo) \to H^*(S^{\bar{n}} \wedge X_5)$ is an isomorphism since the first is spanned by $i_{\bar{n}} \otimes \chi \mathrm{Sq}^{4i}$, $i > 0$. Thus the infinite towers in $\pi_{4i}(S^{\bar{n}} \wedge X_5)$ are mapped in a filtration-preserving way to $\pi_{4i}(B_{\bar{n}} \wedge bo)$. Filling in the diagram, $\pi_{4i}(B_{\bar{n}} \wedge bo) \to \pi_{4i}(B_{\bar{m}} \wedge bo)$ must be filtration preserving. By 3.9 or 3.12, $B_{\bar{n}} \wedge bo$ and $B_{\bar{m}} \wedge bo$ have isomorphic homotopy charts above filtration 0 of the form



where the precise way in which the towers leave filtration 0 depends upon the mod 4 values of $\alpha(\bar{n})$. $\square$

3.7(ii) follows similarly from the observations that if $\bar{m}$ is a 1-successor of $\bar{n}$, then

(1) the homotopy charts of $B_{\bar{n}} \wedge bo$ and $B_{\bar{m}} \wedge bo$ are isomorphic (above filtration 0) until they leave filtration 0, at which point the towers of $B_{\bar{m}} \wedge bo$ always begin one level higher than those of $B_{\bar{n}} \wedge bo$; for example:



(2) the map $S^{\bar{n}} \wedge X_5 \to S^{\bar{m}} \wedge X_5$ lifts to a map $S^{\bar{n}} \wedge X_5 \to S^{\bar{m}} \wedge X_5^{\langle 1 \rangle}$ which induces an isomorphism in $H_*(H^*(\ ); Q_0)$ and hence of infinite homotopy towers. [$H^*(\Sigma X_5^{\langle 1 \rangle}) = \ker(\mathcal{A} \to \{\chi \mathrm{Sq}^{4i}\})$ so that $H_*(H^*(\Sigma X_5^{\langle 1 \rangle}); Q_0)$ is spanned by $\{\mathrm{Sq}^1 \cdot \chi\mathrm{Sq}^{4i}\}$. $H^*(X_5^{\langle 1 \rangle}) \to H^*(X_5)$ sends $\mathrm{Sq}^1 \to i$ and hence $\mathrm{Sq}^1 \cdot \chi\mathrm{Sq}^{4i} = \chi\mathrm{Sq}^{4i} \cdot \mathrm{Sq}^1 + \chi\mathrm{Sq}^{4i-1} \cdot \mathrm{Sq}^2 \mapsto \chi\mathrm{Sq}^{4i}$.]

3.7(iii) follows similarly because if $|\bar{n}| = |\bar{m}|$ and $B_{\bar{n}} \wedge bo \to B_{\bar{m}} \wedge bo$ has filtration 0 or 1, then one of the infinite homotopy towers must be mapped by filtration 0 or 1, and hence the same must be true of $S^{\bar{n}} \wedge X_5 \to S^{\bar{m}} \wedge X_5$. But the above analysis shows this cannot happen unless $\bar{m}$ is a 0- or 1-successor of $\bar{n}$. For example, if $H^*(X_5^{\langle 1 \rangle}) \to H^*(X_5)$ sends $\mathrm{Sq}^1 \to 0$, then all elements of $H_*(H^*(X_5^{\langle 1 \rangle}); Q_0)$ are mapped trivially. If $|\bar{m}| < |\bar{n}|$ any $\mathcal{A}$-homomorphism $H^*(S^{\bar{m}} \wedge X_5) \to H^*(S^{\bar{n}} \wedge X_5)$ must be 0, and similarly for the Q_0-homology elements in $H^*(S^{\bar{m}} \wedge X_5^{\langle 1 \rangle})$.

4. Lifting results. In this section we prove the lifting part of Theorem 1.1. We will use the maps ϕ: $P_{k+8}^{l+8} \to P_k^l$ mentioned in the introduction. They were first constructed in **[8]** and studied in more detail in **[10]**. Because ϕ is defined only when l is even, Theorem 1.1 is somewhat more difficult to prove when n is odd; the case n odd will be considered in Remark 4.4.

The lifting part of Theorem 1.1 when $n \equiv 2$ (8) and $e \equiv 0$ (4) is an immediate consequence of

THEOREM 4.1. *If $e \equiv 0$ (4) and $e \geqslant 20$ and f classifies $a \cdot 2^{e+6}\xi$ with a odd, then for some $d \leqslant 1 + [\log_2(e+2)]/2$ the composite*

$$\Sigma^{-1}P_{2e+11+8d}^{4e+10+8d} \xrightarrow{\phi^d} \Sigma^{-1}P_{2e+11}^{4e+10} \xrightarrow{f} SO \to V_{2e+1}$$

is null-homotopic.

The other cases of Theorem 1.1 when n is even follow from similar results with slight changes in the parameters. (See Remark 4.3.) The parameters here had to satisfy $2^e \cdot a2^{e+6}\xi_{4e+10}$ trivial, $a2^{e+6}\xi_{2e+10}$ trivial, and $(4e+10)+8 < 2(2e+11) - 1$.

PROOF OF 4.1. Let $t = e + 6 - [\log_2(4e+10)]$. By 3.4, f lifts to $SO^{\langle t \rangle}$, and we follow into $V_{2e+1}^{\langle t \rangle}$. By 3.3 if $4e + 10 + 8d \leqslant 2(2e+1) + 2t - 4$, it will suffice to show $\Sigma^{-1}P_{2e+11+8d}^{4e+10+8d} \to^{\phi^d} \Sigma^{-1}P_{2e+11}^{4e+10} \to^{f'} P_{\underline{2e+1}}^{\langle t \rangle} = X$ is null-homotopic. Since $X = P_{\underline{2e+1}}^{\langle t \rangle}$ is Ω^∞ of a stable Adams resolution, it suffices to show the stable map $\Sigma^\infty(f'\phi^d)$ is null-homotopic. The condition simplifies to

$$(*) \qquad 4d \leqslant e - \left[\log_2(4e+10)\right].$$

By the Adams edge theorem [1] for P_{2e+1}, X is at least $(2e - 2 + 2t)$-connected. If X_s denotes the sth stage in the bo-resolution of X, then X_s is at least $(2e - 2 + 2t + 3s)$-connected.

We shall return later to the proof of

LEMMA 4.2. *If l is even, k odd, and $l + 8 < 2k - 1$, ϕ: $P_{k+8}^{l+8} \to P_k^l$ can be written as a composite*

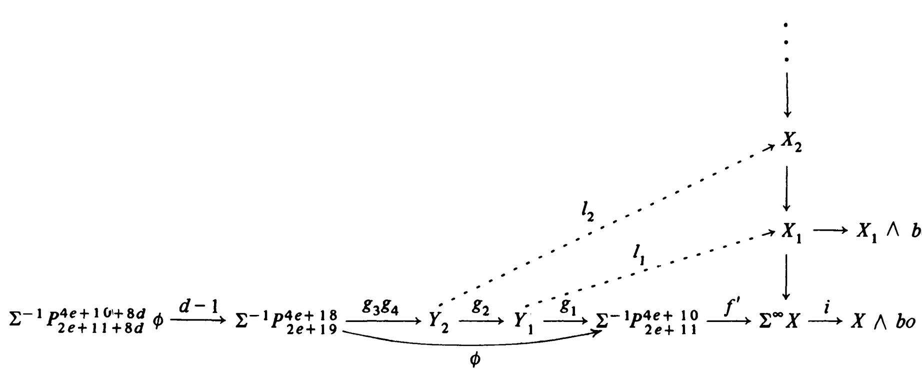
$$P_{k+8}^{l+8} = Y_4 \xrightarrow{g_4} Y_3 \xrightarrow{g_3} Y_2 \xrightarrow{g_2} Y_1 \xrightarrow{g_1} Y_0 = P_k^l$$

such that

(i) $H^*(g_i; \mathbf{Z}_2) = 0$ *for* $1 \leqslant i \leqslant 4$.

(ii) *For* $0 \leqslant i \leqslant 4$ *and X as above, the ASS for $D_N Y_i \wedge X \wedge bo$ has no nontrivial differentials through dimension N.*

In the diagram



where g_1 and g_2 are as in 4.2, we shall show in the next paragraph that the liftings l_1 and l_2 exist. Then 4.2 and 3.6.1 imply that $f' \circ \phi^d$ lifts to X_{4d}, which is at least $(2e - 2 + 2t + 12d)$-connected. If $4d \geqslant 2[\log_2(4e + 10)] - 1$, this connectivity is $\geqslant 4e + 9 + 8d$, and so the map $\Sigma^{-1}P^{4e+10+8d}_{2e+11+8d} \to X_{4d}$ is trivial, and hence so is $f' \circ \phi^d$. Comparing with (*), such a d can be chosen if $4d$ can be chosen in the interval $[2[\log_2(4e + 10)] - 1, e - [\log_2(4e + 10)]]$, and thus can be done if $e \geqslant 20$.

The proof of 4.1 is completed by showing the existence of l_1 and l_2 above. l_1 follows from the observation that any map $Y_1 \to X \wedge bo$ which is cohomologically trivial is null-homotopic. [An associated graded for $[Y_1, X \wedge bo]$ is given by $\mathrm{Ext}^{s,s}_{\mathcal{Q}_1}(H^*X, H^*Y_1)$. By the proof below of 4.2, $DY_1 = (D(\Sigma^{-1}P^{4e+18}_{2e+19}))^{\langle 3\rangle}$ through a sufficiently large skeleton. Thus for $s > 0$,

$$\mathrm{Ext}^{s,s}_{\mathcal{Q}_1}(H^*X, H^*Y_1) \approx \mathrm{Ext}^{s+t-3,s+t-3}_{\mathcal{Q}_1}\big(H^*P_{2e+1}, H^*\big(\Sigma^{-1}P^{4e+18}_{2e+19}\big)\big) = 0$$

by [**6**, 3.4(iii)].]

By 3.7 and 3.9 $X_1 \wedge bo$ splits as $(X \wedge \Sigma^3 bsp) \vee W$, and by 3.6 (and the argument in 3.6.1), since the map $Y_2 \to X_1 \wedge bo$ is cohomologically trivial, l_1 can be varied (if necessary) through $\Omega(X \wedge bo)$ so that the component into W is trivial. A proof similar to that of the previous paragraph shows

$$\big[Y_2, X \wedge \Sigma^3 bsp\big] \overset{(g_3 g_4)^* q_*}{\to} \big[\Sigma^{-1}P^{4e+18}_{2e+19}, P_{2e+1} \wedge \Sigma^3 bsp\big]$$

is injective on cohomologically trivial maps, where q is the usual map $X = P^{\langle t\rangle}_{2e+1} \to P_{2e+1}$. $(g_3 g_4)^* q_*(l_1)$ is a map which when followed into $P_{2e+1} \wedge bJ$ is

$$\Sigma^{-1}P^{4e+18}_{2e+19} \to^{\phi} \Sigma^{-1}P^{4e+10}_{2e+11} \to^{f} SO \to \Omega^{\infty}(P_{2e+1} \wedge bJ),$$

which is trivial since we have chosen these cases so that the bJ-primary obstructions considered in §2 are zero. Since

$$\big[\Sigma^{-1}P^{4e+18}_{2e+19}, P_{2e+1} \wedge \Sigma^3 bsp\big] \to \big[\Sigma^{-1}P^{4e+18}_{2e+19}, P_{2e+1} \wedge bJ\big]$$

is injective above filtration 0, this implies that our map $Y_2 \to X_1 \wedge bo$ is trivial. □

REMARK 4.3. In order to prove the analogue of 4.1 for some of the other congruences it is necessary to use the precise upper edge for $\mathrm{Ext}_{\mathcal{Q}}(H^*P_k, \mathbf{Z}_2)$ rather than the linear approximation to it used in 3.3 and in the estimate of the connectivity of X in the proof of 4.1 (see Remark 3.3.1). Let $D(\alpha, \beta)$ denote the smallest $j > \alpha$ such that $\pi_j(P_\alpha)$ has a nonzero homotopy class of filtration β. Then $D(\alpha, \beta) = \alpha + 2\beta - \varepsilon(\alpha, \beta)$ where

$\varepsilon(\alpha, \beta)$	α (mod 4) ↘ β (mod 4)	0	1	2	3
	0	1	1	2	3
	1	0	1	2	0
α (mod 4)	2	1	1	1	1
	3	0	0	1	2

If we are trying to show $\Sigma^{-1}P_{2e+c+8d}^{4e+b+8d} \to^{\phi^d} \Sigma^{-1}P_{2e+c}^{4e+b} \to^{2^{e+a}\xi} V_{2e+\delta}^{\langle t\rangle}$ is null-homotopic, where $t = e + a - [\log_2(4e + b)]$, then the two conditions on d are

$$4e + b - 1 + 8d < D(2e + \delta, t) + 2e + \delta - 1,$$
$$D(2e + \delta, t) + 12d > 4e + b - 1 + 8d.$$

The parameters a, b, and c are chosen to be the minimal values in the proper congruence satisfying the analogues of the three conditions mentioned after the statement of 4.1.

We tabulate below the values of a, b, and c used in each congruence, the value of δ used (from 1.1), and the smallest value of e for which there is a d satisfying the two inequalities

e mod 4	n mod 8	a	b	c	δ	smallest e
0	0	4	8	9	1	24
0	2	6	10	11	1	12
0	4	7	12	13	1	16
0	6	7	14	13	1	16
1	0	6	12	11	3	17
1	2	8	14	15	0	17
1	4	5	8	9	0	17
1	6	9	18	17	3	17
2	0	8	16	15	2	18
2	2	6	10	13	−1	18
2	4	7	12	13	−1	22
2	6	7	14	13	2	18
3	0	6	12	11	1	23
3	2	8	14	15	−1	19
3	4	5	8	11	−1	23
3	6	9	18	17	1	19

The proof of the analogue of 4.1 for $e = 20$, $n \equiv 0$ (8) requires a slight modification. We use $X = P^{\langle t-1\rangle}{}_{\underline{2e+1}}$ but then can lift to X_1 without using the map g_1 in the factorization of ϕ.

PROOF OF 4.2. $D_N P_k^l \to^{D_N\phi} D_N P_{k+8}^{l+8}$ is a map of filtration 4; in fact it is stably of the same type as ϕ–$\Sigma^{N-L-k}P_{L+k-l}^{L} \to \Sigma^{N-L-k}P_{L+k-l-8}^{L-8}$ for appropriate L. Letting $Z = D_N P_{k+8}^{l+8}$, $D_N\phi$ can be factored as

$$D_N P_k^l \to Z^{\langle 4\rangle} \to Z^{\langle 3\rangle} \to Z^{\langle 2\rangle} \to Z^{\langle 1\rangle} \to Z.$$

If W_i denotes the N-skeleton $(Z^{\langle i\rangle})^{(N)}$, then $D_N\phi$ factors as $D_N P_k^l \to W_3 \to W_2 \to W_1 \to Z$ with each map cohomologically trivial. Applying D_N to this sequence yields the sequence stated in 4.3.

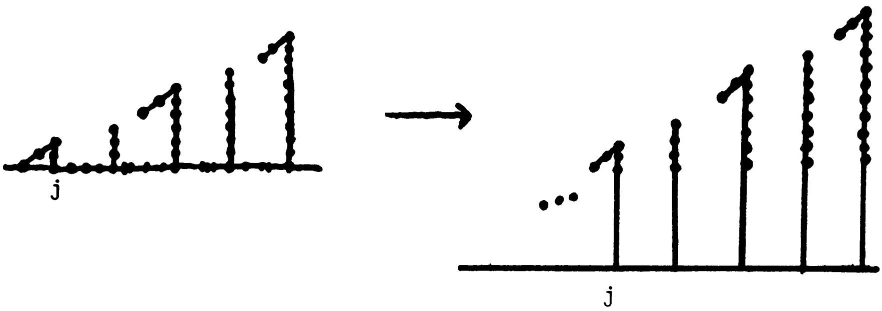
To see that (ii) is satisfied, note that

$$D_N Y_i \wedge X \wedge bo = \left(\left(\Sigma^{N-L-k}P_{L+k-l-8}^{L-8}\right)^{\langle i\rangle}\right)^{(N)} \wedge P_{2e+1}^{\langle t\rangle} \wedge bo = U.$$

For $s > 0$ and $* - s \leqslant N + 2e$,

$$\operatorname{Ext}_{\mathcal{A}}^{s,*}(H^*U, Z_2) \approx \operatorname{Ext}_{\mathcal{A}_1}^{s+i+t,*+i+t}\left(H^*\left(\Sigma^{N-L-k}P_{L+k-l-8}^{L-8} \wedge P_{2e+1}\right), Z_2\right).$$

This was calculated in [**4**, 3.9 and 3.10]. The map in $\operatorname{Ext}_{\mathcal{A}}(\ )$ induced by $U \to \Sigma^{N-L-k}P_{L+k-l-8}^{L-8} \wedge P_{2e+1} \wedge bo$ is through dimension $N + 2e$ of the type suggested by the charts



and hence there can be no differentials in ASS(U) through dimension $N + 2e$. $\square$

REMARK 4.4. The proof of 1.1 when n is odd is complicated by the fact that maps ϕ: $P_{k+8}^{n+8} \to P_k^n$ inducing an isomorphism in K-theory do not exist when n is odd. The lifting can be obtained by using the maps ϕ for P^{n+1}, which is lifted past its bJ-primary obstruction by varying on the top cell.

Consider for example the case $e = 4e'$, $n = 8l + 1$. We must show $\operatorname{gd}(2^{4l+1-4e'}\xi_{8l+1}) = 8e' + 1$ where

$$\Sigma^{-1}P^{8l+2} \xrightarrow{2^{4l+1-4e'}\xi} P_{8e'+1} \wedge bJ$$

has a nonzero obstruction on the top cell. Let $X = \underline{P_{8e'+1}^{\langle t \rangle}}$, where $t = 4l + 1 - 4e' - [\log_2(8l + 2)] - 2$ (two stages lower than maximal). Then f': $\Sigma^{-1}P^{8l+2} \to X$, a lifting of the map which classifies $2^{4l+1-4e'}\xi$, is cohomologically trivial (because f' lifts to $\underline{P_{8e'+1}^{\langle t+1 \rangle}}$). Thus, by the argument in the next-to-last paragraph of the proof of 4.1, $\Sigma^{-1}P^{8l+2} \to^{f'} X \to X \wedge bo$ is trivial so there is a lifting $\Sigma^{-1}P^{8l+2} \to^{l} X_1$. Since f' lifts to $\underline{P_{8e'+1}^{\langle t+2 \rangle}}$, l can be chosen to be 0 in cohomology. As in the last paragraph of the proof of 4.1, l can be chosen so that when followed into $X_1 \wedge bo$ only its component c into $P_{8e'+1}^{\langle t \rangle} \wedge \Sigma^3 bsp$ is nonzero. Since the bJ-primary obstruction for lifting $2^{4l+1-4e'}\xi_{8l+1}$ to $8e' + 1$ is zero (by §2), c is the composite $\Sigma^{-1}P^{8l+2} \to^{p} S^{8l+1} \to^{g} P_{8e'+1} \wedge \Sigma^3 bsp$. Analysis of the upper edge of the Adams charts for

$$\begin{array}{ccc} \pi_*(X_1) & \to & \pi_*(P_{8e'+1} \wedge \Sigma^3 bsp) \\ \downarrow & & \\ \pi_*(\underline{P_{8e'+1}^{\langle t \rangle}}) & \to & \pi_*(P_{8e'+1} \wedge bo) \end{array}$$

shows g is the image of a class $\hat{g} \in \pi_{8l+1}(X_1)$. Let Δ denote the composite $\Sigma^{-1}P^{8l+2} \to^{p} S^{8l+1} \to^{\hat{g}} X_1$. Then $l + \Delta$ lifts to l_2 into X_2 and $l + \Delta | P^{8l+1} = l | P^{8l+1}$. Now use the method of 4.1 to find d such that $\phi^d \circ l_2$ is trivial (for dimensional reasons using the bo-resolution) and is in the range where triviality into $\underline{P}_{8e'+1}^{\langle t\rangle}$ implies triviality into $\underline{V}_{8e'+1}^{\langle t\rangle}$. Thus it suffices to show that

$$\Sigma^{-1}P^{8l+1+8d} \to \Sigma^{-1}P^{8l+2+8d} \xrightarrow{\phi^d} \Sigma^{-1}P_{2m+1}^{8l+2} \xrightarrow{l+\Delta} X_1 \xrightarrow{q} QP_{8e'+1}$$

agrees with $\Sigma^{-1}P^{8l+1+8d} \to SO \to QP_{8e'+1}$ where θ classifies a bundle of order 2^e. This follows since $[\Sigma^{-1}P^{8l+1+8d}, QP_{8e'+1}]$ has no nontrivial elements whose filtration is as large as that of $\Delta \circ \phi^d$.

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